THE FIVE PLATONIC BODIES.

By James H. Weaver.

Pappus, Book III., Propositions 43 to 58, discusses the inscription of the five regular polyhedrons in a sphere.

Taking up the twenty-sided solid, we note that the twelve vertices are in four parallel circles, equal in pairs, the smaller circles circumscribing faces of the icosahedron. The larger circles also contain three vertices of the icosahedron, thus circumscribing larger equilateral triangles.

Pappus proves that:

Diameter of sphere: side of larger triangle: side of smaller triangle.

: : Side of pentagon: side hexagon: side decagon,

these three regular polygons being inscribed in any given circle.

It is now a very easy matter to obtain these equilateral triangles, and construct their circumscribing circles parallel on the sphere, and then to step off on them the vertices of the icosahedron.

Pappus further proves that these same parallel circles contain by fives the twenty vertices of the dodecahedron, the pentagonal face of dodecahedron being inscribed in the self same circle in which the triangular face of the icosahedron was inscribed, a larger pentagon keeps company also with the larger triangle before mentioned, the side of this larger pentagon being the diagonal of the smaller pentagon, and as the square of this line is equal to one third of the square of the diameter of the sphere, it is also the edge of the inscribed cube.

Now a plane passed perpendicular to the diagonal of the smaller pentagon will cut out a circle circumscribing the face of the cube and likewise the face of the octahedron. Moreover, the diameter of this last circle is the edge of the tetrahedron. It is evident that the dodecahedron, the cube, and the tetrahedron can be inscribed in the same sphere with the vertices coinciding. The icosahedron and the octahedron can be obtained

from the dodecahedron and the cube respectively by taking the mid-points of the faces as vertices.

Thus two small circles of the sphere perpendicular to each other on the edge of the pentagon inscribed in the smaller give as sides of regular inscribed polygons the edges of the five regular polyhedrons. The golden section or mean and extreme ratio is particularly prominent in the construction. Note that the connecting link is the edge of the cube which the ancient Greek philosophers called Earth,* the octahedron being water, the tetrahedron fire, the icosahedron air, and the dodecahedron the sphere of the universe. We will note this last contains the key to all the others.

It is to be noticed that not only the mid-points of the twelve faces of the dodecahedron are the twelve vertices of the icosahedron which can be whittled from it, but vice versa there is a

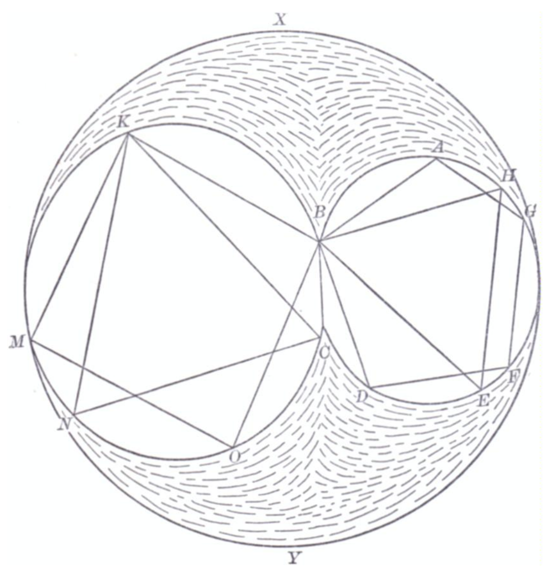

FIG. 1.

reciprocal relation. Now if we take the mid-point of any edge and successively the mid-point of the edge starting from the opposite angle, but not in the same face, we will form from either figure the regular octahedron, the mid-points of whose faces

* "Timæus" of Plato. Translation by Archer Hind, pages 190 seq.

give the cube (there is also a reciprocal relation here), and properly chosen vertices of the cube give the tetrahedron which is invariant.

In addition to these regular solids, Archimedes investigated several others that are equiangular and equilateral. He set up 13 such. They are as follows:

(1) One contained by 8 bases, 4 triangles and 4 hexagons.

Next there are three polyhedra of 14 bases.

(2) The first one of these is contained by 8 triangles and 6 squares.

(3) The second is contained by 8 squares and 8 hexagons.

(4) The third by 8 triangles and 6 octagons.

Then there are two of 26 bases.

(5) The first is contained by 8 triangles and 18 squares.

(6) The second by 12 squares, 8 hexagons and 8 octagons.

Then there are three of 32 bases.

(7) The first is contained by 20 triangles and 12 pentagons.

(8) The second by 12 pentagons and 20 hexagons.

(9) The third by 20 triangles and 12 decagons.

(10) Then there is one of 38 bases contained by 32 triangles and 6 squares.

Then there are two of 62 bases of which

(11) The first is contained by 20 triangles, 30 squares and 12 pentagons.

(12) The second is contained by 30 squares, 20 hexagons and 12 decagons.

(13) The last is of 92 bases, 80 triangles, and 12 decagons.

Now it is interesting to note that all these 13 may be cut from the five regular ones. For example, the first one may be cut from the pyramid by trisecting the edges and cutting off the corners. (2) may be obtained by taking the octahedron, bisecting the edges and then cutting off the six corners. The others may be obtained in the same way from the other solids, by cutting off corners or edges.

This seems to be conclusive evidence that the regular solids were whittled out of the sphere. At least such a thing was possible and a little study of the properties of the figures would lead to such a construction. On the other hand it is not possible to build up the regular solids by combinations of any of the five.

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